

A REMARK ON CHARGE TRANSFER PROCESSES IN MULTI-PARTICLE SYSTEMS

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ABSTRACT. We assess the probability of resonances between sufficiently distant states $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$ in the configuration space of an N -particle disordered quantum system. This includes the cases where the transition $\mathbf{x} \rightsquigarrow \mathbf{y}$ "shuffles" the particles in \mathbf{x} , like the transition $(a, a, b) \rightsquigarrow (a, b, b)$ in a 3-particle system. In presence of a random external potential $V(\cdot, \omega)$ (Anderson-type models) such pairs of configurations (\mathbf{x}, \mathbf{y}) give rise to local (random) Hamiltonians which are strongly coupled, so that eigenvalue (or eigenfunction) correlator bounds are difficult to obtain (cf. [4], [11]). This difficulty, which occurs for $N \geq 3$, results in eigenfunction decay bounds weaker than expected. We show that more efficient bounds, obtained so far only for 2-particle systems [11], extend to any $N > 2$.

1. INTRODUCTION. THE MODEL

We study quantum systems in a disordered environment, usually referred to as Anderson-type models, due to the seminal paper by P. Anderson [6]. For nearly *fifty years* following its publication, the localization phenomena have been studied in the single-particle approximation, i.e. under the assumption that the interaction between particles subject to the common disordered (usually understood as "random") external potential is "sufficiently weak" to be neglected in the analysis of the decay properties of eigenstates of the multi-particle system in question. A detailed discussion of recent developments in the physics of disordered media is most certainly beyond the scope of this paper; we simply refer to recent papers by Basko–Aleiner–Altshuler [7] and by Gornyi–Mirlin–Polyakov [21] (the order of citation is merely alphabetical) where it was shown, in the framework of physical models and methods, that the localization phenomena, firmly established in the non-interacting multi-particle disordered quantum systems, persist in presence of non-trivial interactions.

The rigorous mathematical analysis of localization in strongly disordered quantum systems started approximately "Twenty Years After" the aforementioned Anderson's paper, when Goldsheid and Molchanov [19] first proved the existence of point spectrum of the Sturm–Liouville operator

$$H_V = -\frac{d^2}{dt^2} + V(t, \omega), \quad t \in \mathbb{R},$$

with random potential of the form $V(t, \omega) = F(X_t(\omega))$ generated by a sample of sufficiently regular Markov process $X_\bullet(\omega)$ and a "non-flat" smooth function F on some auxiliary phase space, and then the complete spectral localization was established for such operators by Goldsheid, Molchanov and Pastur [20].

A few years later, Kunz and Souillard [24] proved a similar result (spectral localization) for a lattice ("tight-binding") Anderson model in one dimension.

Techniques of [19, 20, 24] are applicable only to one-dimensional systems, or, with some modifications, to "quasi-one-dimensional" models, like "tubes" $\mathcal{D} \times \mathbb{R} \subset \mathbb{R}^{d+1}$, $\mathcal{D} \subset \mathbb{R}^d$, $d \geq 1$, extended only in one dimension. For this reason, further progress in higher-dimensional Anderson-type models has been made only several years later, approximately "Thirty Years After" the publication of Anderson's original paper, in the works by Fröhlich and Spencer [16], Holden and Martinelli [22], Fröhlich, Martinelli, Scoppola and Spencer [17]. Later, Spencer [27] gave a *very short* and simple proof of exponential decay of the Green functions (i.e. the kernel of the resolvent $G(x, y; E) = (H - E)^{-1}(x, y)$). Combined with a result by Simon and Wolff [26], this implied spectral localization. The MSA procedure was reformulated by von Dreifus and Klein [15], giving rise not only to a simplified finite-volume criterion of localization, but also to a versatile approach which has later been adapted to various models, including those in the Euclidean space \mathbb{R}^d , $d \geq 1$, both for standard Schrödinger operators $H_V = -\Delta + V$ and for more general second-order differential operators. A detailed discussion and an extensive bibliography can be found, e.g., in the monograph [28]. A survey of a multitude of techniques and results concerning Anderson-type models in the Euclidean space \mathbb{R}^d , $d \geq 1$, is also beyond the scope of the present paper. We do not discuss either an elegant alternative approach to the localization developed by Aizenman and Molchanov [2] (see also [1, 3]), and recently adapted by Aizenman and Warzel [4, 5] to multi-particle tight binding models.

For the sake of clarity of the presentation, we consider the Hamiltonian $\mathbf{H}_{V, \mathbf{U}}(\omega)$ in the Hilbert space $\mathcal{H}_N := \ell^2(\mathbb{Z}^{Nd})$, of the form

$$\mathbf{H}_{V, U} = \sum_{j=1}^N \left(\Delta^{(j)} + V(x_j, \omega) \right) + \mathbf{U}, \quad (1.1)$$

where $V : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}$ is a random field relative to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\Delta^{(j)} \Psi(x_1, \dots, x_N) = \sum_{y \in \mathbb{Z}^d: |y - x_j| = 1} \Psi(x_1, \dots, x_j + y, \dots, x_N),$$

and \mathbf{U} is the multiplication operator by a function $\mathbf{U}(\mathbf{x})$ which we assume bounded (this assumption can be relaxed). The symmetry of the function \mathbf{U} is not required, and we do not assume \mathbf{U} to be a "short-range" or rapidly decaying interaction. In fact, we focus here on restrictions of $\mathbf{H}_{V, \mathbf{U}}$ to finite subsets of the lattice, so that $\mathbf{H}_{V, \mathbf{U}}$ may or may not be well-defined on the entire lattice \mathbb{Z}^d : this does not affect our main result.

The assumptions on the random field V are described below, in Section 3.

1.1. Eigenvalue concentration bounds. We focus here on probabilistic bounds of certain eigenvalue correlators, or eigenvalue concentration bounds, known in the single-particle localization theory as Wegner-type bounds, due to the paper by Wegner [29]. It would not be an exaggeration to say that this bound is the heart of the MSA. (In a slightly disguised form, it also appears in the framework of the FMM both in its the single-particle and multi-particle version, as the reader can observe in [2, 4, 5].) In essence, one may call a Wegner-type bound a (sufficiently explicite and suitable for applications) probabilistic bound of the form

$$\mathbb{P} \{ \text{dist}(E, \sigma(\mathbf{H}_\Lambda(\omega))) \leq \epsilon \} \leq f(|\Lambda|, \epsilon), \quad (\text{W1})$$

where $\mathbf{H}_\Lambda(\omega)$ is the restriction of $\mathbf{H}(\omega)$ on a bounded subset Λ with some self-adjoint boundary conditions, and $\sigma(\mathbf{H}_\Lambda(\omega))$ is its spectrum (a finite number of random points, in the case of lattice models).

The role and importance of such bounds can be easily understood: the MSA procedure starts with the analysis of the resolvents $(\mathbf{H}_\Lambda - E)^{-1}$, so it is vital to know how unlikely it is to have the spectrum of \mathbf{H}_Λ ϵ -close to a given value $E \in \mathbb{R}$.

Unfortunately, the Wegner-type bound alone, in its original form given in (W1), does not suffice in the context of multi-particle (even two-particle) models. It does not matter, actually, how sharp is the "one-volume" bound of the type (W1): it simply does not provide a sufficient input for multi-particle adaptations of the MSA. For this reason, we do not discuss here a series of very interesting papers including [12], [14], [13]. The knowledge of the so-called limiting integrated density of states (IDS, in short) of a multi-particle system is also insufficient for the multi-particle MSA (MPMSA). Such information is indeed available due to Klopp and Zenk [25] who proved that the limiting IDS of an N -particle system with a decaying interaction between particles is the same as without interaction.

In the case where the marginal probability distribution function F_V of an IID random field V is analytic in a strip around the real axis (e.g., Gaussian, Cauchy) a Wegner-type bound was proven in our earlier work [8] where we also proved the analyticity of finite-volume eigenvalue distributions. Kirsch [23] proved an analog of the finite-volume bound (W1), with an optimal volume dependence, for multi-particle systems under the assumption of bounded marginal density. Zenk [30] established a Wegner-type bound for multi-particle systems in \mathbb{R}^d , $d \geq 1$, with a realistic long-range particle-particle interaction. However, it should be emphasized again that a one-volume Wegner-type bound (W1), *even if it were an exact, explicite equality* and not just an upper bound, seems so far **insufficient** for the MPMSA to work. On the other hand, bounds of the form (W1) **are** necessary for the MPMSA, but for this purpose it suffices to have the volume dependence very far from optimal. For instance, f may have the form

$$f(|\Lambda|, \epsilon) = C|\Lambda|^B \ln^{-A} \epsilon, \quad (1.2)$$

with any given $B < +\infty$ and sufficiently large $A > 0$. A stronger bound with $f(|\Lambda|, \epsilon) = C|\Lambda|^B \epsilon^b$, $b > 0$, which can usually be obtained under assumption of Hölder-continuity of the marginal distribution of an IID random field $V(x, \omega)$, is more than sufficient. The reader can see that it is much weaker than "the" Wegner bound with $f(|\Lambda|, \epsilon) = C|\Lambda| \epsilon$.

Given any finite cube $\mathbf{C}_L(\mathbf{u}) := \{\mathbf{x} \in \mathbb{Z}^{Nd} \mid |\mathbf{x} - \mathbf{u}| \leq L\}$, we will consider a finite-volume approximation of the Hamiltonian \mathbf{H}

$$\mathbf{H}_{\mathbf{C}_L(\mathbf{u})} = \mathbf{H} \upharpoonright_{\ell^2(\mathbf{C}_L(\mathbf{u}))} \text{ with Dirichlet boundary conditions on } \partial \mathbf{C}_L(\mathbf{u}).$$

acting in the finite-dimensional Hilbert space $\ell^2(\mathbf{C}_L(\mathbf{u}))$. In [9] the following "two-volume" version of the Wegner bound was established for pairs of two-particle operators $\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}$, $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}$ such that $L \geq L'$ and $\text{dist}(\mathbf{C}_L(\mathbf{u}), \mathbf{C}_{L'}(\mathbf{u}')) \geq 8L$: if ν is the continuity modulus of the marginal distribution of the IID random field V , then

$$\mathbb{P} \left\{ \text{dist}(\sigma(\mathbf{H}_{\mathbf{C}_L(\mathbf{u})}), \sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')})) \leq \epsilon \right\} \leq (2L + 1)^{2d} (2L' + 1)^d \nu(2\epsilon). \quad (\text{W2})$$

The proof given in [9] is based on a geometrical notion of "separable" pairs of cubes, combined with Stollmann's lemma on "diagonally monotone" functions. In [8] a similar bound was proven in the case of IID random field V with analytic marginal distribution.

Unfortunately, starting from $N = 3$, additional difficulties appear in the analysis of pairs of spectra $\sigma(\mathbf{H}_{\mathbf{C}_L(\mathbf{u})})$, $\sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}))$. To put it simply, no a priori lower bound

on the distance $\text{dist}(\mathbf{C}_L(\mathbf{u}), \mathbf{C}_L(\mathbf{u}')) > CL$ between two cubes of sidelength $O(L)$ can guarantee the approach of [9] to work, no matter how large is the constant C . This gives rise to a significantly more sophisticated MPMSA procedure in the general case where $N \geq 3$. A similar difficulty arises in [4].

1.2. The main goal. It is well-known that the FMM, when applicable, leads directly to the proof of the dynamical localization, while it is more natural for applications of the MSA to establish first the spectral localization, via probabilistic bounds of the kernels of resolvents $G_\Lambda(E) = (H_\Lambda - E)^{-1}$ in finite subsets (usually cubes) $\Lambda \subset \mathbb{Z}^d$, and then derive dynamical localization from decay bounds of the resolvents $G_\Lambda(E)$.

In [10, 11] a multi-particle adaptation of the MSA was used to prove *spectral* localization (i.e., exponential decay of eigenfunctions) in the strong disorder regime. Aizenman and Warzel [4, 5] used the FMM to prove directly *dynamical* localization (hence, spectral localization) in various parameter regions including strong disorder, "extreme" energies and weak interactions.

Despite many differences between these two approaches, similar technical difficulties have been encountered in both cycles of papers. Namely, it turned out to be difficult to prove the decay bounds of eigenfunctions $\Psi_j^{(N)}(x_1, \dots, x_N)$ of N -particle Hamiltonians in terms of some *norm* $\|\cdot\|$ in \mathbb{R}^{Nd} :

$$|\Psi_j^{(N)}(x_1, \dots, x_N; \omega)| \leq C_j(\omega) e^{-m\|\mathbf{x}\|}.$$

If the interaction \mathbf{U} is symmetric (and so is, then, $\mathbf{U} + \mathbf{V}$), then it is natural to expect (or to fear ...) "resonances" and "tunneling" between a point $\mathbf{x} = (x_1, \dots, x_N)$ and the points $\tau(\mathbf{x}) = (x_{\tau(1)}, \dots, x_{\tau(N)})$ obtained by permutations $\tau \in \mathfrak{S}_N$. So, it is much more natural in this context to use the "symmetrized" distance

$$d_S(\mathbf{x}, \mathbf{y}) := \min_{\tau \in \mathfrak{S}_N} \|\mathbf{x} - \mathbf{y}\|.$$

Note also that if the quantum particles are bosons or fermions, then the points $\tau(\mathbf{x})$ should even be treated as identical, or, more precisely, the spectral problem should be solved in the subspace of symmetric or anti-symmetric functions of variables x_j .

However, due to a highly correlated nature of the potential of a multi-particle system, even the above concession did not suffice, and it was easier to use "Hausdorff distance" (see the definition below, in Section 2) between points $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{Nd}$. This resulted in weaker decay estimates than expected. (Note that the Hausdorff distance was not used directly in [11].)

Aizenman and Warzel [4] analyzed explicitly the aforementioned technical problem and pointed out that, physically speaking, it was difficult to rule out the possibility of "tunneling" between points \mathbf{x} and \mathbf{y} related by a "partial charge transfer" process, e.g., between points (a, a, b) and (a, b, b) , $a \neq b$, corresponding to the states:

$$\begin{aligned} \text{state } \mathbf{x}: & \text{ 2 particles at the point } a \text{ and 1 particle at } b \\ \text{state } \mathbf{y}: & \text{ 1 particle at the point } a \text{ and 2 particles at } b. \end{aligned}$$

Observe that the norm-distance between such states can be arbitrarily large.

In the present paper we address this problem and show that resonances between distant states in the configuration space, related by partial charge transfer processes, are unlikely, providing probabilistic estimates for such unlikely situations.

1.3. Structure of the paper.

- The main theorem of this paper is formulated in Section 1.4.
- In Section 2 we introduce basic notions and give a geometrical sufficient condition for "weak separability" of multi-particle configurations (see Lemma 2.3).
- In Section 3.1 we formulate and prove the main technical result, Lemma 3.1.
- The assertion of Theorem 1 follows from Lemma 3.1 combined with Lemma 2.3; the proof (quite short) is given in Section 3.2.

1.4. The main result. Introduce the following notations. Given a parallelepiped $Q \subset \mathbb{Z}^d$, we denote by $\xi_Q(\omega)$ the sample mean of the random field V over the Q ,

$$\xi_Q(\omega) = \langle V \rangle_Q = \frac{1}{|Q|} \sum_{x \in Q} V(x, \omega)$$

and the "fluctuations" of V relative to the sample mean,

$$\eta_x = V(x, \omega) - \xi_Q(\omega), \quad x \in Q.$$

We denote by \mathfrak{F}_Q the sigma-algebra generated by $\{\eta_x, x \in Q\}$, and by $F_\xi(\cdot | \mathfrak{F}_Q)$ the conditional distribution function of ξ_Q given \mathfrak{F}_Q :

$$F_\xi(s | \mathfrak{F}_Q) := \mathbb{P} \{ \xi_Q \leq s | \mathfrak{F}_Q \}.$$

For a given $s \in \mathbb{R}$, $F_\xi(s | \mathfrak{F}_Q)$ is a random variable, determined by the values of $\{\eta_x, x \in Q\}$, but we will often use inequalities involving it, meaning that these relations hold true for \mathbb{P} -a.e. condition.

We will assume that the random field V satisfies the following condition¹ (CCM = Continuity of the Conditional Mean):

(CCM): *There exist constants $C', C'', A', A'', b', b'' \in (0, +\infty)$ such that $\forall Q \subset \mathbb{Z}^d$ with $\text{diam}(Q) \leq R$ the conditional distribution function $F_\xi(\cdot | \mathfrak{F}_Q)$ satisfies for all $s \in (0, 1)$*

$$\mathbb{P} \left\{ \sup_{t \in \mathbb{R}} |F_\xi(t + s | \mathfrak{F}_Q) - F_\xi(t | \mathfrak{F}_Q)| \geq C' R^{A'} s^{b'} \right\} \leq C'' R^{A''} s^{b''}. \quad (1.3)$$

Note that in the case of an IID random field V the probability distribution of ξ_Q is the same for all finite subsets Q of given cardinality $|Q|$. In the particular case of a Gaussian IID field V , e.g., with zero mean and unit variance, ξ_Q is a Gaussian random variable with variance $|Q|^{-1}$, independent of the "fluctuations" η_x , so that its probability density is bounded:

$$p_{\xi_Q}(s) = |Q|^{1/2} (2\pi)^{-1/2} e^{-\frac{|Q|s^2}{2}} \leq |Q|^{1/2} (2\pi)^{-1/2},$$

although the L_∞ -norm of its probability density grows as $|Q| \rightarrow \infty$, and so does the continuity modulus of the distribution function F_{ξ_Q} .

Theorem 1. *Let $V : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}$ be a random field satisfying (CCM). Then for any pair of N -particle operators $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')} , \mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')}$, $0 \leq L', L'' \leq L$, satisfying $d_S(\mathbf{u}', \mathbf{u}'') > 2(N+1)L$, and any $s > 0$ the following bound holds:*

$$\mathbb{P} \{ \text{dist}(\sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{u}')}), \sigma(\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{u}'')})) \leq s \} = h_L(s) \quad (1.4)$$

¹In an earlier version of this manuscript (1005.3387v2, 02.07.2010), we assumed a stronger condition: a uniform continuity of the conditional probability distribution function $F_\xi(\cdot | \mathfrak{F}_Q)$, i.e., a uniform bound for a.e. condition.

with

$$h_L(s) := |\mathbf{C}_{L''}(\mathbf{x})| \cdot |\mathbf{C}_{L''}(\mathbf{y})| C' L^{A'} s^{b'} + C'' L^{A''} s^{b''}. \quad (1.5)$$

In general, the conditional distribution function $F_\xi(\cdot | \mathfrak{F}_Q)$ is not necessarily uniformly continuous, let alone Hölder-continuous. Moreover, the following simple example shows that for some conditions the distribution of the sample mean can be extremely singular.

Example 1.1. Let $v_1(\omega), v_2(\omega)$ be two independent random variables uniformly distributed in $[0, 1]$. Set $\xi = (v_1 + v_2)/2$, $\eta = (v_1 - v_2)/2$. Conditioning on $\eta \geq 0$ induces a uniform probability distribution on the segment $I(\eta) = \{(t + 2\eta, t), t \in (0, 1 - 2\eta)\}$ of length $|I(\eta)| = 1 - 2\eta$, with constant probability density $(1 - 2\eta)^{-1}$, if $\eta < 1/2$. Obviously, these distributions are not uniformly continuous. Moreover, for $\eta = 1/2$, ξ takes a single value: $\xi = 1/2$, so that its conditional distribution is no longer continuous. Observe, however, that "singular" conditions have probability zero, and conditions which give rise to large conditional density of ξ have small probability.

Using the main idea of the above example, Gaume [18], in the framework of his PhD project, proved the property **(CCM)** for the IID potential with a uniform marginal distribution, and later extended the proof to IID potentials with piecewise constant marginal probability density.

Our estimate is certainly *not sharp* and can probably be improved in various ways. Nevertheless, it suffices for the purposes of the MPMSA and allows to substantially simplify its structure, while leading to stronger results.

Finally, note that we consider here only lattice models for the sake of brevity and clarity of presentation.

2. CONFIGURATIONS AND WEAK SEPARABILITY

2.1. Basic definitions. Consider the lattice $(\mathbb{Z}^d)^N \cong \mathbb{Z}^{Nd}$, $N > 1$. We denote by \mathbb{D} the "principal diagonal" in $(\mathbb{Z}^d)^N$:

$$\mathbb{D} = \{\mathbf{x} \in \mathbb{Z}^{Nd} : \mathbf{x} = (x, \dots, x), x \in \mathbb{Z}^d\}.$$

Intervals of integer values will often appear in our formulae, and it is convenient to use a standard notation $[[a, b]] := [a, b] \cap \mathbb{Z}$.

We identify vectors $\mathbf{x} \in \mathbb{Z}^{Nd}$ with configurations of N distinguishable particles in \mathbb{Z}^d : $\mathbf{x} \equiv (x_1, \dots, x_N) \in \mathbb{Z}^d \times \dots \times \mathbb{Z}^d$. Working with norms of vectors $\mathbf{x} \in \mathbb{Z}^{Nd}$, we always use the canonical embedding $\mathbb{Z}^{Nd} \subset \mathbb{R}^{Nd}$.

Definition 2.1. Let $\mathbf{x} \in \mathbb{Z}^{Nd}$ be an N -particle configuration and consider a subset $\mathcal{J} \subset [[1, N]]$ with $1 \leq |\mathcal{J}| = n < N$. A **subconfiguration** of \mathbf{x} associated with \mathcal{J} is the pair $(\mathbf{x}', \mathcal{J})$ where the vector $\mathbf{x}' \in \mathbb{Z}^{nd}$ has the components $x'_i = x_{j_i}$, $i \in [[1, n]]$. Such a subconfiguration will be denoted as $\mathbf{x}_{\mathcal{J}}$. The complement of a subconfiguration $\mathbf{x}_{\mathcal{J}}$ is the subconfiguration $\mathbf{x}_{\mathcal{J}^c}$ associated with the complementary index subset $\mathcal{J}^c := [[1, N]] \setminus \mathcal{J}$.

By a slight abuse of notations, we will identify a subconfiguration $\mathbf{x}_{\mathcal{J}} = (\mathbf{x}', \mathcal{J})$ with the vector \mathbf{x}' . With \mathcal{J} clearly identified (this will always be the case in our arguments), it should not lead to any ambiguity, while making notations simpler.

Definition 2.2. (a) Let $N \geq 2$ and consider the set of all N -particle configurations \mathbb{Z}^{Nd} . For each $j \in [[1, N]]$ the coordinate projection $\Pi_j : \mathbb{Z}^{Nd} \rightarrow \mathbb{Z}^d$ onto the coordinate space of the j -th particle is the mapping

$$\Pi_j : (x_1, \dots, x_N) \mapsto x_j.$$

(b) The support $\Pi \mathbf{x}$ of a configuration $\mathbf{x} \in \mathbb{Z}^{nd}$, $n \geq 1$, is the set

$$\Pi \mathbf{x} := \cup_{j=1}^n \Pi_j \mathbf{x} = \{x_1, \dots, x_N\}.$$

Similarly, the support of a subconfiguration $\mathbf{x}_{\mathcal{J}}$ is defined by $\Pi \mathbf{x}_{\mathcal{J}} := \cup_{j \in \mathcal{J}} \Pi_j \mathbf{x}$.

(c) Given a subset $\mathcal{J} \subset [[1, N]]$ with $|\mathcal{J}| = n$, the projection $\Pi_{\mathcal{J}} : \mathbb{Z}^{Nd} \rightarrow \mathbb{Z}^d$ is defined as follows:

$$\Pi_{\mathcal{J}} \mathbf{x} = \begin{cases} \Pi \mathbf{x}_{\mathcal{J}}, & \text{if } \mathcal{J} \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

Finally, for each subset $\mathbf{C} \subset \mathbb{Z}^{Nd}$ its support $\Pi \mathbf{C}$ is defined by

$$\Pi \mathbf{C} := \bigcup_{j=1}^N \Pi_j \mathbf{C} \subset \mathbb{Z}^d.$$

We will not associate with the empty subconfiguration \mathbf{x}_{\emptyset} any object other than its support $\Pi \mathbf{x}_{\emptyset} = \emptyset \subset \mathbb{Z}^d$, so the above definitions and notations suffice for our purposes.

It is worth noticing that we use the max-norm $\|\cdot\|_{\infty}$ in \mathbb{R}^{nd} , $n \geq 1$,

$$\|\mathbf{x}\|_{\infty} = \|(x_1, \dots, x_n)\|_{\infty} := \max_{j \in [[1, n]]} \max_{i \in [[1, d]]} |x_j^{(i)}|.$$

It is often more suitable for multi-particle geometric bounds than the Euclidean norm. This norm canonically induces the notion of diameter, denoted below as "diam".

Particle configurations being associated with point subsets of \mathbb{Z}^d , one can introduce the distance between two arbitrary configurations $\mathbf{x}' \in \mathbb{Z}^{N'd}$, $\mathbf{x}'' \in \mathbb{Z}^{N''d}$, $N', N'' \geq 1$, as the distance between the respective subsets of \mathbb{Z}^d , induced by the max-norm:

$$\begin{aligned} \rho(\mathbf{x}', \mathbf{x}'') &\equiv \rho(\{x'_1, \dots, x'_{N'}\}, \{x''_1, \dots, x''_{N''}\}) \\ &= \min_{i \in [[1, N']]} \min_{j \in [[1, N'']]} \|x'_i - x''_j\|. \end{aligned}$$

Further, given a bounded subset $\mathcal{X} \subset \mathbb{R}^d$, we call the canonical envelope of \mathcal{X} the minimal parallelepiped \mathcal{Q} with edges parallel to coordinate hyperplanes and such that $\mathcal{X} \subseteq \mathcal{Q}$. It will often be denoted as $\mathcal{Q}(\mathcal{X})$. It is readily seen that $\text{diam}(\mathcal{X}) = \text{diam}(\mathcal{Q}(\mathcal{X}))$ (recall that "diam" is induced by the max-norm). Keeping in mind the canonical embedding $\mathbb{Z}^d \subset \mathbb{R}^d$, this notion applies also to bounded lattice subsets.

Now we introduce a modified version of the distance ρ which allows to simplify certain geometrical constructions used below; it suffices for our purposes to define it for bounded subsets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$ as follows:

$$\rho_{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = \rho(\mathcal{Q}(\mathcal{X}), \mathcal{Q}(\mathcal{Y})),$$

where $\mathcal{Q}(\mathcal{X})$ and $\mathcal{Q}(\mathcal{Y})$ are the canonical envelopes of sets \mathcal{X} and, respectively, \mathcal{Y} . (Here \mathcal{C} in $\rho_{\mathcal{C}}$ stands for "convex").

For further references, we state the following elementary

Lemma 2.1. *Consider bounded subsets $\mathcal{X}, \mathcal{Y} \subset \mathbb{Z}^d \subset \mathbb{R}^d$. Then:*

- (A) $\rho_{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \geq R \geq 1 \Leftrightarrow$ there exist two hyperplanes $\mathcal{L}', \mathcal{L}'' \subset \mathbb{Z}^d$, parallel to one of the coordinate hyperplanes, at max-norm distance R from each other and such that the layer with the border $\mathcal{L}' \cup \mathcal{L}''$ separates \mathcal{X} from \mathcal{Y} ;
- (B) if $\rho_{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = 0$, then $\text{diam}(\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})) \leq \text{diam}(\mathcal{X}) + \text{diam}(\mathcal{Y})$.

We would like to stress that the distance ρ_C , unlike the distance ρ , does *not always* separate disjoint sets. In particular, the assertion (B) can (and will) be applied to some disjoint pairs of sets, too.

The proof is straightforward, so we omit it.

Definition 2.3. A particle configuration $\mathbf{x} \in \mathbb{Z}^{Nd}$, $N > 1$, is called R -decoupled, with $R > 0$, if it contains a pair of non-empty complementary subconfigurations $\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}$ with $\rho_C(\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}) > 2R$. The decoupling width of a configuration \mathbf{x} is the quantity

$$D(\mathbf{x}) := \max_{\mathcal{J}, \mathcal{J}^c \subseteq [1, N]} \rho_C(\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}).$$

The decoupling width $D(\mathbf{C}_L(\mathbf{x}))$ of a cube $\mathbf{C}_L(\mathbf{x})$ is defined as follows:

$$D(\mathbf{C}_L(\mathbf{x})) := \max_{\mathcal{J}, \mathcal{J}^c \subseteq [1, N]} \rho_C \left(\bigcup_{i \in \mathcal{J}} C_L(x_i), \bigcup_{j \in \mathcal{J}^c} C_L(x_j) \right).$$

In other words, R -decoupling of \mathbf{x} means that R -neighborhoods of two complementary (and non-empty) subconfigurations $\mathbf{x}_{\mathcal{J}}, \mathbf{x}_{\mathcal{J}^c}$ are disjoint.

Definition 2.4. An R -cluster of a configuration \mathbf{x} is a subconfiguration $\mathbf{x}_{\mathcal{J}}$ which is not R -decoupled and is not contained in any non- R -decoupled subconfiguration $\mathbf{x}_{\mathcal{J}'}$ with $|\mathcal{J}'| > |\mathcal{J}|$. The set of all R -clusters of a configuration \mathbf{x} is denoted by $\Gamma(\mathbf{x}, R)$.

Lemma 2.2. Fix a positive number R and integers $N > 1$, $d \geq 1$.

- (A) Every configuration $\mathbf{x} \in \mathbb{Z}^{Nd}$ can be decomposed into a family $\Gamma(\mathbf{x}, R)$ of R -clusters $\Gamma_1, \dots, \Gamma_M$, with some $1 \leq M = M(\mathbf{x}) \leq N$, so that $(\mathcal{J}_{\Gamma_1}, \dots, \mathcal{J}_{\Gamma_M})$ is a partition of $[1, N]$ and each particle x_j is contained in exactly one cluster $\Gamma = \Gamma_{\mathbf{x}}(x_j)$.
- (B) If $\Gamma', \Gamma'' \in \Gamma(\mathbf{x}, R)$ and $\Gamma' \neq \Gamma''$, then $\rho_C(\Gamma', \Gamma'') > 2R$, i.e. the canonical envelopes obey $\rho(\mathcal{Q}(\Gamma'), \mathcal{Q}(\Gamma'')) > 2R$.
- (C) The diameter of any cluster Γ is bounded by $2(N-1)R$, and so is the diameter of its envelope $\mathcal{Q}(\Gamma)$.

Proof. It is convenient here to make use of the canonical embedding $\mathbb{Z}^d \subset \mathbb{R}^d$. For each $i \in [1, N]$ introduce the cube $C'_R(x_i) \subset \mathbb{R}^d$. The union $\bigcup_i C'_R(x_i)$ is uniquely decomposed into a union of maximal ρ_C -disjoint components $\{\tilde{C}_j\}$: for some partition $\{\mathcal{J}_j, j \in [1, M]\}$ of $[1, N]$ with $1 \leq M \leq N$,

$$\bigcup_{i=1}^N C'_R(x_i) = \prod_{j=1}^M \left(\bigcup_{i \in \mathcal{J}_j} C'_R(x_i) \right) =: \prod_{j=1}^M \tilde{C}_j,$$

and for all $j \neq j'$, $\mathcal{Q}(\tilde{C}_j), \mathcal{Q}(\tilde{C}_{j'})$ obey

$$\rho(\mathcal{Q}(\tilde{C}_j), \mathcal{Q}(\tilde{C}_{j'})) > 0. \quad (2.1)$$

Set

$$\Gamma_j = \tilde{C}_j \cap \Pi \mathbf{x} \equiv \tilde{C}_j \cap \{x_1, \dots, x_N\} \subset \mathbb{Z}^d, \quad j = 1, \dots, M.$$

Observe that (2.1) implies

$$\rho_C(\Gamma_{j'}, \Gamma_{j''}) \geq \rho_C(\tilde{C}_{j'}, \tilde{C}_{j''}) + 2R > 2R.$$

Finally, $\text{diam}(\Gamma_j) \leq 2(|\Gamma_j| - 1)R$: it suffices to combine the assertion (B) of Lemma 2.1, the fact that \tilde{C}_j is a union of cubes of diameter $2R$, and the inequality $|\Gamma_j| \leq N$. \square

2.2. Weakly separable cubes.

Definition 2.5. A cube $\mathbf{C}_L(\mathbf{x})$ is weakly separable from $\mathbf{C}_L(\mathbf{y})$ if there exists a parallelepiped $Q \subset \mathbb{Z}^d$ in the 1-particle configuration space, of diameter $R \leq 2NL$, and subsets $\mathcal{J}_1, \mathcal{J}_2 \subset [[1, N]]$ such that $|\mathcal{J}_1| > |\mathcal{J}_2|$ (possibly, with $\mathcal{J}_2 = \emptyset$) and

$$\begin{aligned} \Pi_{\mathcal{J}_1} \mathbf{C}_L(\mathbf{x}) \cup \Pi_{\mathcal{J}_2} \mathbf{C}_L(\mathbf{y}) &\subseteq Q, \\ \Pi_{\mathcal{J}_2^c} \mathbf{C}_L(\mathbf{y}) \cap Q &= \emptyset. \end{aligned} \quad (2.2)$$

A pair of cubes $(\mathbf{C}_L(\mathbf{x}), \mathbf{C}_L(\mathbf{y}))$ is weakly separable if at least one of the cubes is weakly separable from the other.

The physical meaning of the weak separability is that in a certain region of the one-particle configuration space the presence of particles from configuration \mathbf{x} is more important than that of the particles from \mathbf{y} . As a result, some local fluctuations of the random potential $V(\cdot; \omega)$ have a stronger influence on \mathbf{x} than on \mathbf{y} .

Some upper bound on the diameter R of the "separating parallelepiped" Q is required in applications of the main theorem to the localization analysis of multi-particle systems. Indeed, the continuity modulus ν_L figuring in the main bound (1.4) depends upon the size $L \geq L', L''$, and it grows with L . However, in most cases there is a substantial freedom for the choice of the upper bound of $\text{diam } Q$; cf. (1.2). Again, the case of a Gaussian IID random potential V is instructive here.

Lemma 2.3. Cubes $\mathbf{C}_L(\mathbf{x}), \mathbf{C}_L(\mathbf{y})$ with $d_S(\mathbf{x}, \mathbf{y}) > 2(N+1)L$ are weakly separable.

Proof. Let $\Gamma(\mathbf{x}, 2L)$ and $\Gamma(\mathbf{y}, 2L)$ be the collections of $2L$ -clusters for the configurations \mathbf{x} and \mathbf{y} . Consider clusters $\Gamma_1, \dots, \Gamma_M \in \Gamma(\mathbf{x}, 2L)$, $M = |\Gamma(\mathbf{x}, 2L)|$, and the disjoint parallelepipeds

$$Q_i = \mathcal{Q} \left(\bigcup_{k: x_k \in \Gamma_i} C_L(x_k) \right), \quad i \in [[1, M]]. \quad (2.3)$$

By virtue of assertion (C) of Lemma 2.2,

$$\text{diam}(Q_i) \leq \text{diam}(\Gamma_i) + 2L \leq 2(N-1)L + 2L = 2NL.$$

Introduce the "occupancy numbers" of parallelepipeds Q_i for configurations \mathbf{x} and \mathbf{y} :

$$\begin{aligned} n_i(\mathbf{x}) &= \text{card}(\Pi \mathbf{x} \cap Q_i), \quad i \in [[1, M]], \\ n_i(\mathbf{y}) &= \text{card}(\Pi \mathbf{y} \cap Q_i), \quad i \in [[1, M]]. \end{aligned}$$

There can be two possible situations:

- (I) For all $i \in [[1, M]]$ we have $n_i(\mathbf{x}) = n_i(\mathbf{y})$. Then there exists a permutation $\tau \in \mathfrak{S}_N$ such that for all $j \in [[1, N]]$,

$$\|x_{\tau(j)} - y_j\| \leq 2(N-1)L + 2L = 2NL,$$

yielding

$$d_S(\mathbf{x}, \mathbf{y}) \leq \|\tau(\mathbf{x}) - \mathbf{y}\| = \max_{1 \leq j \leq N} \|x_{\tau(j)} - y_j\| \leq 2NL.$$

If $d_S(\mathbf{x}, \mathbf{y}) > 2(N+1)L$, then the occupancy numbers $n_i(\mathbf{x}), n_i(\mathbf{y})$ cannot be all identical, so this situation is impossible under the hypotheses of the lemma.

- (II) For some $i \in [[1, M]]$, $n_i(\mathbf{x}) \neq n_i(\mathbf{y})$. By the definition (2.3) of Q_i , it contains $|\Gamma_i| \geq 1$ particles of the configuration \mathbf{x} , so that $n_i(\mathbf{x}) \geq 1$ for all $i \in [[1, M]]$. Observe that

$$\sum_{i=1}^M (n_i(\mathbf{x}) - n_i(\mathbf{y})) = N - \sum_{i=1}^M n_i(\mathbf{y}) \geq 0. \quad (2.4)$$

Since not all quantities $n_i(\mathbf{x}) - n_i(\mathbf{y})$ vanish, there exists some $j_o \in [[1, M]]$ such that $n_{j_o}(\mathbf{x}) - n_{j_o}(\mathbf{y}) > 0$, otherwise the LHS of (2.4) would be negative.

Now setting $Q = Q_{j_o}$, we see that the conditions (2.2) are fulfilled. \square

3. EIGENVALUE CONCENTRATION BOUND FOR DISTANT CUBES

3.1. Bounds for weakly separable cubes.

Lemma 3.1. *Let $V : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$ be a random field satisfying the condition (CCM). Let $\mathbf{x}, \mathbf{y} \in \mathcal{Z}$ be two configurations such that the cubes $\mathbf{C}_L(\mathbf{x})$, $\mathbf{C}_L(\mathbf{y})$ are weakly separable. Consider operators $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{y})}(\omega)$, $\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{y})}(\omega)$, with $L', L'' \leq L$. Then for any $s > 0$ the following bound holds for the spectra $\Sigma_{\mathbf{x}} = \sigma(\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{x})})$, $\Sigma_{\mathbf{y}} = \sigma(\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{y})})$ of these operators:*

$$\mathbb{P} \{ \text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s \} \leq |\mathbf{C}_L(\mathbf{u}')| |\mathbf{C}_L(\mathbf{u}'')| C' L^{A'} (2s)^{b'} + C'' L^{A''} (2s)^{b''}.$$

Proof. Let Q be a ball satisfying the conditions (2.2) for some $\mathcal{J}_1, \mathcal{J}_2 \subset [[1, N]]$ with $|\mathcal{J}_1| = n_1 > n_2 = |\mathcal{J}_2|$. Introduce the sample mean $\xi = \xi_Q$ of V over Q and the fluctuations $\{\eta_x, x \in Q\}$ defined as in Section 1.4.

The operators $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{x})}(\omega)$, $\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{y})}(\omega)$ read as follows:

$$\begin{aligned} \mathbf{H}_{\mathbf{C}_{L'}(\mathbf{x})}(\omega) &= n_1 \xi(\omega) \mathbf{1} + \mathbf{A}(\omega), \\ \mathbf{H}_{\mathbf{C}_{L''}(\mathbf{y})}(\omega) &= n_2 \xi(\omega) \mathbf{1} + \mathbf{B}(\omega) \end{aligned} \quad (3.1)$$

where operators $\mathbf{A}(\omega)$ and $\mathbf{B}(\omega)$ are \mathfrak{F}_Q -measurable. Let $\{\lambda_1, \dots, \lambda_{M'}\}$, $M' = |\mathbf{C}_{L'}(\mathbf{x})|$, and $\{\mu_1, \dots, \mu_{M''}\}$, $M'' = |\mathbf{C}_{L''}(\mathbf{y})|$, be the sets of eigenvalues of $\mathbf{H}_{\mathbf{C}_{L'}(\mathbf{x})}$ and of $\mathbf{H}_{\mathbf{C}_{L''}(\mathbf{y})}$, counted with multiplicities. Owing to (3.1), these eigenvalues can be represented as follows:

$$\lambda_j(\omega) = n_1 \xi(\omega) + \lambda_j^{(0)}(\omega), \quad \mu_j(\omega) = n_2 \xi(\omega) + \mu_j^{(0)}(\omega),$$

where the random variables $\lambda_j^{(0)}(\omega)$ and $\mu_j^{(0)}(\omega)$ are \mathfrak{F}_Q -measurable. Therefore,

$$\lambda_i(\omega) - \mu_j(\omega) = (n_1 - n_2) \xi(\omega) + (\lambda_i^{(0)}(\omega) - \mu_j^{(0)}(\omega)),$$

with $n_1 - n_2 \geq 1$, owing to our assumption. Further, we can write

$$\begin{aligned} \mathbb{P} \{ \text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s \} &= \mathbb{P} \{ \exists i, j : |\lambda_i - \mu_j| \leq s \} \\ &\leq \sum_{i=1}^{M'} \sum_{j=1}^{M''} \mathbb{E} \{ \mathbb{P} \{ |\lambda_i - \mu_j| \leq s \mid \mathfrak{F}_Q \} \}. \end{aligned}$$

Note that for all i and j we have

$$\begin{aligned} \mathbb{P} \{ |\lambda_i - \mu_j| \leq s \mid \mathfrak{F}_Q \} &= \mathbb{P} \left\{ |(n_1 - n_2) \xi + \lambda_i^{(0)} - \mu_j^{(0)}| \leq s \mid \mathfrak{F}_Q \right\} \\ &\leq \nu_L(2|n_1 - n_2|^{-1}s \mid \mathfrak{F}_Q) \leq \nu_L(2s \mid \mathfrak{F}_Q). \end{aligned}$$

Consider the event

$$\mathcal{E}_L = \left\{ \sup_{t \in \mathbb{R}} |F_\xi(t + s | \mathfrak{F}_Q) - F_\xi(t | \mathfrak{F}_Q)| \geq C' L^{A'} s^{b'} \right\}$$

By hypothesis **(CCM)**(cf. (1.3)), $\mathbb{P} \{ \mathcal{E}_L \} \leq C'' L^{A''} s^{b''}$. Therefore,

$$\begin{aligned} \mathbb{P} \{ \text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s \} &= \mathbb{E} \{ \mathbb{P} \{ \text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s | \mathfrak{F}_Q \} \} \\ &\leq \mathbb{E} \{ \mathbf{1}_{\mathcal{E}_L} \mathbb{P} \{ \text{dist}(\Sigma_{\mathbf{x}}, \Sigma_{\mathbf{y}}) \leq s | \mathfrak{F}_Q \} \} + \mathbb{P} \{ \mathcal{E}_L \} \\ &\leq |\mathbf{C}_{L''}(\mathbf{x})| \cdot |\mathbf{C}_{L''}(\mathbf{y})| C' L^{A'} s^{b'} + C'' L^{A''} s^{b''} = h_L(s) \end{aligned}$$

with h_L defined in (1.5). \square

3.2. Proof of the main result.

By the hypothesis of Theorem 1, we have $d_S(\mathbf{x}, \mathbf{y}) > 2(N+1)L$; therefore, by Lemma 2.3, cubes $\mathbf{C}_{L'}(\mathbf{x})$ and $\mathbf{C}_{L''}(\mathbf{y})$ are weakly separable. Now the assertion of the theorem follows from Lemma 3.1. \square

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